Euler-Lagrange correspondence of generalized Burgers cellular automaton

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Abstract

Recently, we have proposed a *Euler-Lagrange transformation* for cellular automata(CA) by developing new transformation formulas. Applying this method to the Burgers CA(BCA), we have succeeded in obtaining the Lagrange representation of the BCA. In this paper, we apply this method to multi-value generalized Burgers CA(GBCA) which include the Fukui-Ishibashi model and the quick-start model associated with traffic flow. As a result, we have succeeded in clarifying the Euler-Lagrange correspondence of these models. It turns out, moreover that the GBCA can naturally be considered as a simple model of a multi-lane traffic flow.

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INTRODUCTION

Cellular automata(CA) are simple objects that can describe not only physical systems but also complex systems such as chemical, biological and social systems. Since Wolfram has introduced the elementary CA (ECA) in one-dimension as a model of complex system[1], lots of studies have been done in these fields.

Among them, conservative CA[2, 3, 4], including soliton systems[5, 6] and traffic-flow models[7, 8], have attracted much attention because of their analytical properties which can be treated by various mathematical methods.

In recent years, the authors have proposed and studied the so-called ultra-discrete method by which we obtain corresponding CA from difference equations[9, 10, 11, 12, 13, 14]. Various soliton equations have been transformed into CA by this method and the solutions of the CA have also been obtained exactly without losing the mathematical properties of corresponding difference equations. In this method, max-function plays an important role through the ultra-discrete formula,

$$\lim_{\epsilon\to+0} \epsilon \log\left(\exp\left(\frac{A}{\epsilon}\right)+\exp\left(\frac{B}{\epsilon}\right)\right)=\max(A,B). \tag{1}$$

On the other hand, it is known that several conservative CA allow two different representation, *Euler* representation and *Lagrange* representation[15]. In the Euler representation, particles are observed at a certain fixed point in space as a dependent(field) variable, while in the Lagrange representation, we trace each particle and follow the trajectory of it. Thus a dependent variable represents the position of each particle in the Lagrange representation. In the previous paper[16], we have proposed a Euler-Lagrange transformation for CA by developing new explicit transformation formulas. We started from Burgers CA(BCA), which is the Euler representation of rule-184 CA,

$$U_j^{t+1}=U_j^t+\min(U_{j-1}^t,1-U_j^t)-\min(U_j^t,1-U_{j+1}^t), \tag{2}$$

where U_j^t denote the number of particles at the site j and time t and introducing the variable S by

$$S_j^t=\sum_{k=-\infty}^{j} U_k^t, \tag{3}$$

or $U_j^t = S_j^t - S_{j-1}^t$, and putting

$$S_j^t = \sum_{i=0}^{N-1} H(j - x_i^t), \tag{4}$$

where $H(x)$ is the step function defined by $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ otherwise, we obtain Lagrange representation of rule-184,

$$\begin{aligned} x_i^{t+1} &= \min(x_i^t + 1, x_{i+1}^t - 1) \\ &= x_i^t + \min(1, x_{i+1}^t - x_i^t - 1), \end{aligned} \tag{5}$$

where x_i^t is the Lagrange variable that represents the position of the i-th particle at time t.

In deriving the Lagrange representation from Euler representation, following two formulae for max function and step function play important roles.

One is

$$\sum_{k=1}^{n} H(j - \min(a_k, b_k)) = \max\left(\sum_{k=1}^{n} H(j - a_k), \sum_{k=1}^{n} H(j - b_k)\right), \tag{6}$$

where we assume that $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$. This formula expresses the commutability of max function and step function. Another formula is

$$\max\left(\sum_i H(j - a_i^t) - m, 0\right) = \sum_i H(j - a_{i+m}^t), \tag{7}$$

where we assume that $a_j^t = \infty$ if j is larger than the number of particles.

In the Euler-Lagrange transformation, max-function again plays an important role as in the ultra-discrete method. In addition, the step function significantly contributes the transformation as well.

GENERALIZED BURGERS CELLULAR AUTOMATON

In this paper, we present a new model which is a multi-value (multi-lane) and multi-neighbor (high-speed, long-perspective) extension of BCA. The equation in the Euler representation is expressed by

$$U_j^{t+1} - U_j^t = \min\left(\sum_{k=0}^{V-1} U_{j-1-k}^t,\ \sum_{k=0}^{P-1}(L - U_{j+k}^t)\right) - \min\left(\sum_{k=0}^{V-1} U_{j-k}^t,\ \sum_{k=0}^{P-1}(L - U_{j+1+k}^t)\right), \tag{8}$$

where $U_j^t \in \{0, \cdots, L\}$ denotes the number of particles at the site j and time t. The parameter L represents the maximum capacity of a cell, V represents the maximum speed of particles and P represents perspective of particles, i.e., the maximum number of particles that a particle can see in its front. We call this model generalized Burgers cellular automaton (GBCA) in this paper. It will become clear later that GBCA include Euler representation of Fukui-Ishibashi model[7] and quick-start model[17, 18], which are both known as traffic models.

Let us derive the Lagrange representation of GBCA according to the method which we have proposed in the previous paper[16]. First we introduce the variable S by

$$S_j^t = \sum_{k=-\infty}^{j} U_k^t, \tag{9}$$

or $U_j^t = S_j^t - S_{j-1}^t$ where S_j^t is the total number of the particles from $-\infty$ to j-th site, which is assumed to be finite. Rewriting (8) in S, we obtain

$$S_j^{t+1} = \max(S_{j-V}^t, S_{j+P}^t - LP). \tag{10}$$

Here, we put

$$S_j^t = \sum_{i=0}^{N-1} H(j - x_i^t), \tag{11}$$

where $H(x)$ is the step function and N is the total number of particles on the cells. x_i^t is the Lagrange variable that represents the position of the i-th particle at time t and the relation $x_0^t < x_1^t < \cdots < x_{N-1}^t$ holds.

Using (11) to replace S in (10) by H, we have

$$\sum_{i=0}^{N-1} H(j - x_i^{t+1}) = \max\left(\sum_{i=0}^{N-1} H(j - x_i^t - V), \sum_{i=0}^{N-1} H(j - x_i^t + P) - LP\right). \tag{12}$$

By using (6) and (7), we have

$$\begin{aligned} \sum_i H(j - x_i^{t+1}) &= \max\left(\sum_i H(j - x_i^t - V), \sum_i H(j - x_{i+LP}^t + P)\right) \\ &= \sum_i H\left(j - \min(x_i^t + V, x_{i+LP}^t - P)\right), \end{aligned} \tag{13}$$

Comparing both sides, we finally obtain

$$\begin{aligned} x_i^{t+1} &= \min(x_i^t + V, x_{i+LP}^t - P) \\ &= x_i^t + \min(V, x_{i+LP}^t - x_i^t - P) \end{aligned} \tag{14}$$

This is the Lagrange representation of GBCA. If we put $L = 1, P = 1$, (14) becomes the Lagrange representation of the Fukui-Ishibashi model[7],

$$x_i^{t+1} = x_i^t + \min(V, x_{i+1}^t - x_i^t - 1). \tag{15}$$

Therefore we know the Euler representation of Fukui-Ishibashi model is

$$U_j^{t+1} - U_j^t = \min\left(\sum_{k=0}^{V-1} U_{j-1-k}^t,\ 1 - U_j^t\right) - \min\left(\sum_{k=0}^{V-1} U_{j-k}^t,\ 1 - U_{j+1}^t\right). \tag{16}$$

If we put $L = 1, V = 1$, we obtain the Lagrange representation of the quick-start model[17, 18],

$$x_i^{t+1} = x_i^t + \min(1, x_{i+P}^t - x_i^t - P). \tag{17}$$

Corresponding Euler representation of the quick-start model is

$$U_j^{t+1} - U_j^t = \min\left(U_{j-1}^t,\ \sum_{k=0}^{P-1}(1 - U_{j+k}^t)\right) - \min\left(U_j^t,\ \sum_{k=0}^{P-1}(1 - U_{j+1+k}^t)\right). \tag{18}$$

The expression in the case of $P = 2$ has already been obtained in [18]. Thus (18) is a generalization of the previous result to the arbitrary P.

MULTI-VALUE (MULTI-LANE) CA

In the previous section, we show that (8) in the case of $L = 1$ includes the Euler representation of the Fukui-Ishibashi model and the quick-start model, and clarify the Euler-Lagrange correspondence of these models.

In this section, we show that (8) in the case of $L > 1$ can be interpreted as an multi-value (multi-lane) extension of the Fukui-Ishibashi model or the quick-start model.

Let us consider (8) in the case of $P = 1$ for simplicity. The Euler representation becomes

$$U_j^{t+1} - U_j^t = \min\left(\sum_{k=0}^{V-1} U_{j-1-k}^t,\ L - U_j^t\right) - \min\left(\sum_{k=0}^{V-1} U_{j-k}^t,\ L - U_{j+1}^t\right), \tag{19}$$

and the corresponding Lagrange representation becomes

$$x_i^{t+1} = x_i^t + \min(V, x_{i+L}^t - x_i^t - 1). \tag{20}$$

We show in the following that (19) is a $(L+1)$-valued CA and (20) describes the movement of particle in L-lane, which means (19) and (20) are considered as a $(L+1)$-valued (L-lane) extension of Fukui-Ishibashi model.

Assume that $L > 0$ and $0 \le U_j^t \le L$ for any j at a certain t. Then, relations

$$
\begin{aligned}
\min\left(\sum_{k=0}^{V-1} U_{j-1-k}^t,\ L - U_j^t\right) &\ge 0 \\
\min\left(\sum_{k=0}^{V-1} U_{j-k}^t,\ L - U_{j+1}^t\right) &\ge 0 \\
\min\left(\sum_{k=0}^{V-1} U_{j-1-k}^t,\ L - U_j^t\right) + U_j^t &= \min\left(\sum_{k=-1}^{V-1} U_{j-1-k}^t,\ L\right) \le L \\
\min\left(\sum_{k=0}^{V-1} U_{j-k}^t,\ L - U_{j+1}^t\right) - U_j^t &= \min\left(\sum_{k=1}^{V-1} U_{j-k}^t,\ L - U_j^t - U_{j+1}^t\right) \\
&\le \min\left(\sum_{k=0}^{V-1} U_{j-1-k}^t,\ L - U_j^t\right)
\end{aligned}
\tag{21}
$$

holds. Therefore, $0 \le U_j^{t+1} \le L$ holds for any j. This means the equation (19) under the above condition is equivalent to a CA with a value set $\{0, 1, \ldots, L\}$.

In the case of $L = 2$ and $V = 2$, the time evolution of (19) is illustrated by the FIG. 1, where the number of the particles in the jth cell is equals to the value of U_j^t. We can

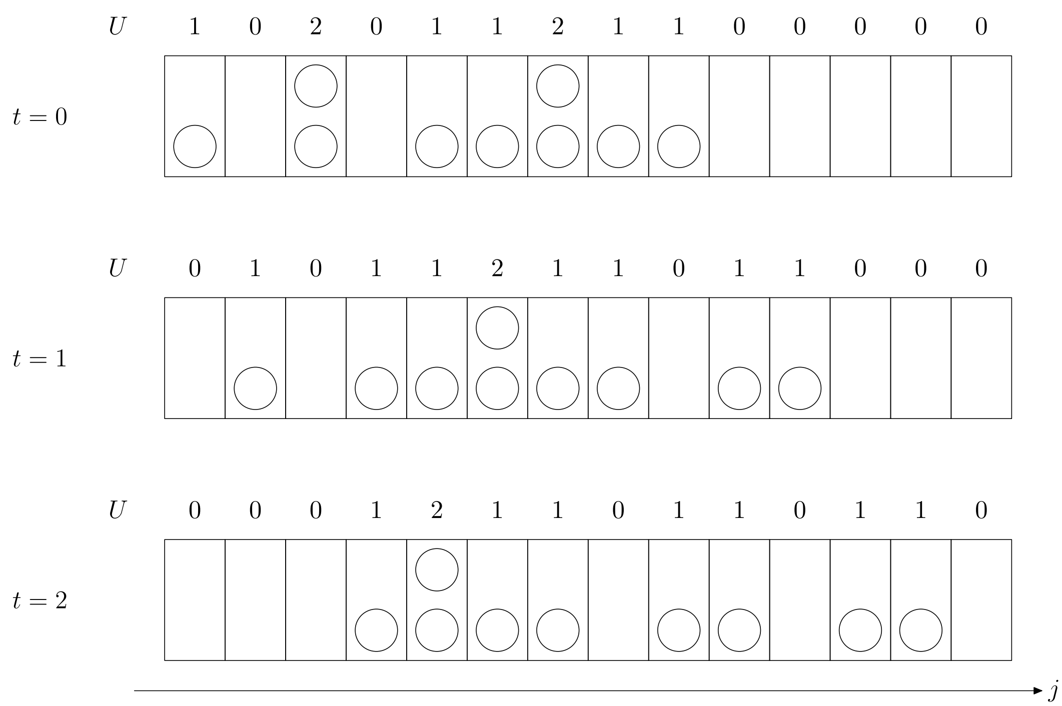

FIG. 1: time evolution of the equation (19)

consider this case as a 3-valued CA in the Euler representation.

Next, we introduce an ordering line on cells as shown in FIG. 2 . We number particles along this line as shown in FIG.3, where i denotes suffix of variable x_i^t in (20). For example, particles of the initial condition in FIG. 1 are numbered as FIG. 4.

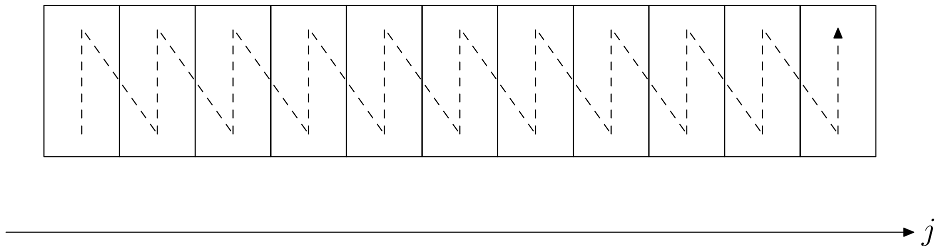

FIG. 2: ordering line

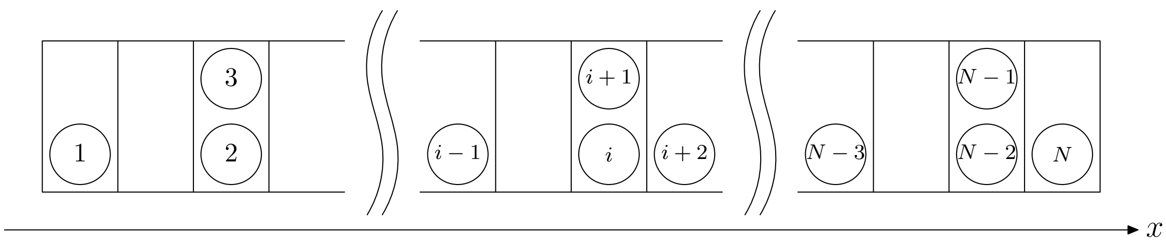

FIG. 3: numbered particles along ordering line

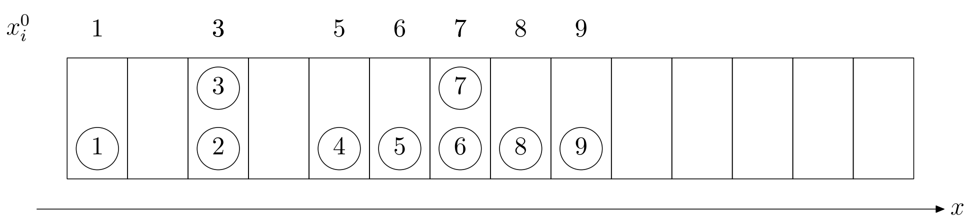

FIG. 4: numbered particles at $t = 0$ of FIG.1

If we take FIG. 4 as a initial condition for (20) and move particles according to (20), we obtain FIG. 5. The configurations of particles in FIG. 5 is identical with FIG. 1 if we neglect numbers assigned to particles. Note here that although (20) is a form of a car following model, the exclusion principle in one cell, which means $x_i \neq x_j$ holds for any i, j , does not hold any more. Actually we see $x_4^1 = x_5^1$ surely holds in FIG 5. Without the exclusion principle, (20) works successfully together with (19), and describe the movement

of the particles. Note that there are no overtaking particles in the course of time on the ordering line. As the the number of particle in one cell remains less than 2, the model is considered as a car following model in 2-lane in this case.

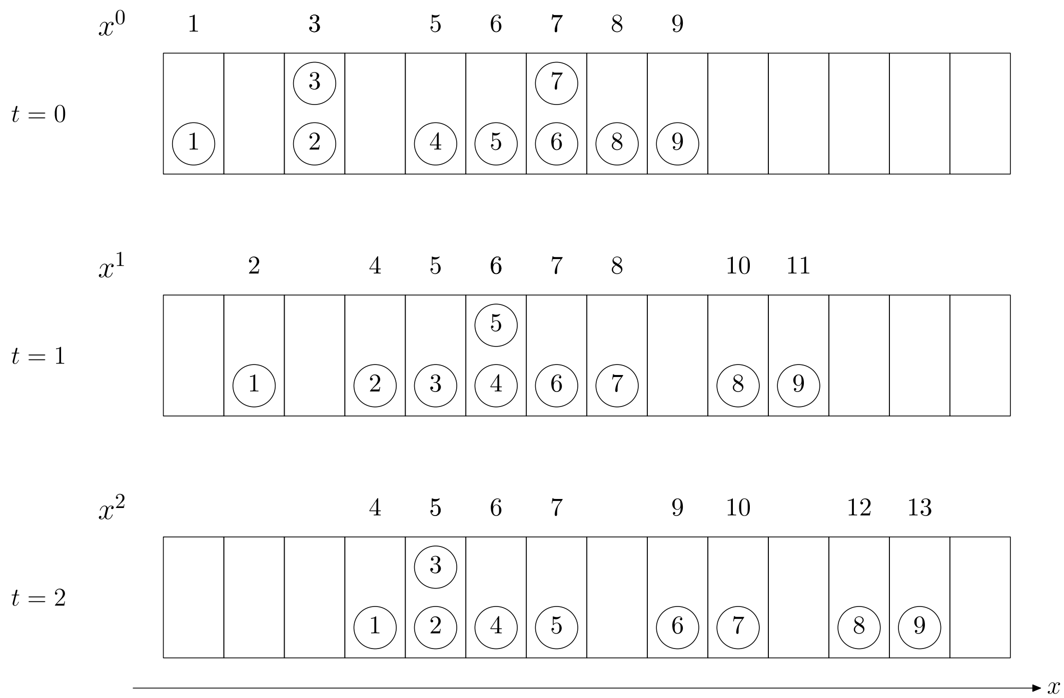

FIG. 5: time evolution of the equation (20)

Generally, as (19) is $(L+1)$-valued CA and (20) is a car-following form of equation like the Fukui-Ishibashi model, we can consider (19) and (20) as $(L+1)$-valued (L-lane) extension of Fukui-Ishibashi model.

In the case of the quick-start model, we can similarly construct multi-value (multi-lane) extension of the model although we don't discuss in detail in this paper.

FUNDAMENTAL DIAGRAMS

In this section, we discuss about fundamental diagrams of GBCA. In the following, we consider a periodic boundary condition. GBCA can be expressed in a conservation form such as

$$\Delta_t U_j^t + \Delta_j q_j^t = 0, \tag{22}$$

where Δ_t and Δ_j are the forward difference operator with respect to the indicated variable and q_j^t represents a flow. Average density ρ and average flow Q^t over all sites are defined by

$$\rho \equiv \frac{1}{KL}\sum_{j=1}^{K} U_j^t, \qquad Q^t \equiv \frac{1}{KL}\sum_{j=1}^{K} q_j^t \tag{23}$$

where K is the number of sites in a period. Since GBCA is in a conservation form, average density does not depend on time and we can use ρ without a superscript t.

As we have shown, GBCA can be expressed by (10). As (10) can be transformed into ultra-discrete diffusion type equation, asymptotic behavior of GBCA is similar to that of BCA. Since a pattern of the maximum flow is given by $\cdots \underbrace{11\cdots 11}_{P}\underbrace{00\cdots 00}_{V}\cdots$, the density and the flow at the phase transition from free to congested state become $\dfrac{P}{P+V}$ and $\dfrac{PV}{P+V}$, respectively[15]. Thus the fundamental diagram is given by FIG. 6.

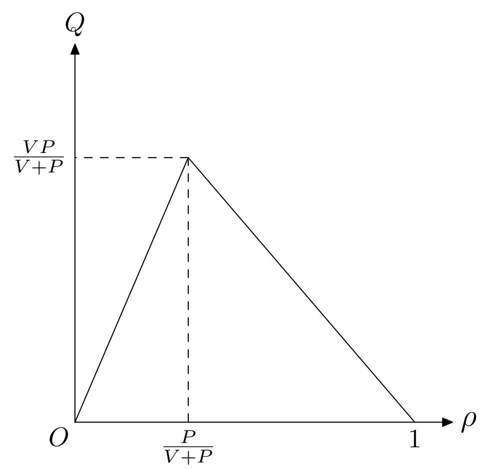

FIG. 6:

CONCLUDING DISCUSSIONS

In this paper, we have proposed GBCA which is a multi-value (multi-lane) and multi-neighbor (high-speed, long-perspective) extension of BCA. We have derived the Lagrange representation of GBCA, which includes the Fukui-Ishibashi model and the quick-start model when $L = 1$. By using the Euler-Lagrange correspondence for GBCA, we have also obtained the Euler representation of Fukui-Ishibashi model. Furthermore, we have obtained multi-value (multi-lane) extension of the Fukui-Ishibashi model and the quick-start model, and shown the multi-lane interpretation by introducing ordering line on cells. It is worth mentioning again here that the success of the Euler-Lagrange transformation in the multi-lane case is due to the relaxation of the exclusion principle in (20).

We have applied our method to more general cases than the case in the previous papers and it works successfully. There are more several conservative CA model which allows both Euler and Lagrange representation. For examples, soliton systems called box and ball system[5] are known to allow two different representation. Now we are progressing on finding the Euler-Lagrange correspondence for soliton systems and shall report them in the future.

We believe that establishing the general Euler-Lagrange correspondence of CA will make a new development of the ultra-discrete method and studies of CA.

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